Ex 3. : What is the probability that a leap year selected at random will contain 53 wednesdays? W.B.U. Tech. 2002, 2007

Soln : A leap year contains 366 days. 366 days = 52 full weeks + two days (i) Monday and Tuesday (ii) Tuesday and Wednesday (iii) Wednesday and Thrusday (iv) Thrusday and Friday (v) Friday and Saturday (vi) Saturday and Sunday (vii) Sunday and Monday.

Therefore a leapyear will contain 53 Wednesday if one of the extra two days are Wednesdays.

∴ the required probability

$$= \frac{\text{Favourable no of cases}}{\text{Total no of cases}}$$

$$= \frac{2}{7}$$

Ex 4. : If A and B are two events such that $P(A) = P(B) = 1$, then show that $P(A \cup B) = 1$ and $P(A \cap B) = 1$

Soln : $P(A \cup B) = P(A) + P(B) - P(A \cap B)$

$= 1 + 1 - P(A \cap B)$

$= 2 - P(A \cap B)$ ....(i)

Now $P(A \cap B) \leq 1$

$\Rightarrow -P(A \cap B) \geq -1$

$\Rightarrow 2 - P(A \cap B) \geq 2 - 1 = 1$

$\Rightarrow P(A \cup B) \geq 1$

But $P(A \cup B) \leq 1$

$\therefore \quad P(A \cup B) = 1$

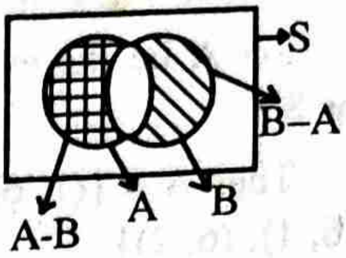
$P(A \cup B) = 2 - P(A \cap B)$

$\Rightarrow P(A \cap B) = 2 - P(A \cup B)$

$= 2 - 1 = 1$

Ex 5. : Show that the probability of occurence of only one of the events A and B is $P(A) + P(B) - 2P(A \cap B)$

Sol : The event that the occurence of only one of the events A and B is



$$(A - B) \cup (B - A)$$

$$= (A \cap B^c) \cup (B \cap A^c)$$

Now $P\{(A \cap B^c) \cup (B \cap A^c)\}$

$$= P(A \cap B^c) + P(B \cap A^c) \text{ .... (i)}$$

[$\because$ $(A \cap B^c)$ and $(B \cap A^c)$ are mutually exclusive

Now $A = (A \cap B^c) \cup (A \cap B)$

$\therefore$ $P(A) = P(A \cap B^c) + P(A \cap B)$

[$\because$ $(A \cap B^c)$ and $(A \cap B)$ are mutually exclusive events]

$\therefore$ $P(A \cap B^c) = P(A) - P(A \cap B)$

similarly $P(A^c \cap B) = P(B) - P(A \cap B)$

putting these values in (1) we get

$$P\{(A \cap B^c) \cup (B \cap A^c)\} = P(A) - P(A \cap B) + P(B) - P(A \cap B)$$

$$= P(A) + P(B) - 2P(A \cap B) \text{ Proved.}$$

Ex 6. : A bag contain 9 white and 4 black balls. If 6 balls are drawn at random, what is the probability that 3 are white and 3 are black.

Sol : 6 balls can be chosen from 13 balls in ${}^{13}c_6$ ways.

$\therefore$ Total number of cases = ${}^{13}c_6$

3 white balls can be chosen from 9 white balls in ${}^{9}c_3$ ways.

3 balcks balls can be chosen from 4 blacks balls in ${}^{4}c_3$ ways.

$\therefore$ favourable number of cases = ${}^{9}c_3 \times {}^{4}c_3$

$\therefore$ the required probability $= \dfrac{{}^{9}c_3 \times {}^{4}c_3}{{}^{13}c_6}$

$$= \frac{28}{143}$$

Ex 7. : When two dice are thrown, find the probability that the sum of the points on the dice is 7 or 8.

Ans. Here sample space is

S = {(1, 1), (1, 2), (1, 3), (1, 4), (1, 5), (1, 6), (2, 1), (2, 2), (2, 3), (2, 4), (2, 5), (2, 6), (3, 1), (3, 2), (3, 3), (3, 4), (3, 5), (3, 6), (4, 1), (4, 2), (4, 3), (4, 4), (4, 5), (4, 6), (5, 1), (5, 2), (5, 3), (5, 4), (5, 5), (5, 6), (6, 1), (6, 2), (6, 4), (6, 5), (6, 6)}

Ex 12. : The number 1, 2, ...,n are arranged in random order. What is the probability that the numbers 1 and 2 are always together?

Soln : n numbers can be arranged in $\lfloor n$ ways.

$\therefore$ total number of cases $\lfloor n$.

Total number of cases where 1 and 2 are always together is $\lfloor 2 \times \lfloor n-1$

$$\therefore \text{ the required probability} = \frac{\lfloor 2 \times \lfloor n-1}{\lfloor n}$$

$$= \frac{2}{n}$$

Ex 13. : The integers x andy are choosen at random with replacement from the set of natural numbers {1, 2, ..., 9}. Find the probability that $|x^2 - y^2|$ is divisible by 2.

Soln : Two numbers x and y can be chooscn at random with replacement from 9 numbers in 9^2 ways.

$\therefore$ total numbner of cases = 9^2

$|x^2 - y^2|$ will be divisible by 2 if x and y are both even or x and y are both odd.

Let A be the event that "x and y are both even"

and B be the event that "x and y are both odd"

NowA contains 4^2 event points

B contains 5^2 event points

$\therefore P(A \cup B) = P(A) + P(B)$ $[\because A \cap B = \phi]$

Now $P(A) = \frac{4^2}{9^2}$, $P(B) = \frac{5^2}{9^2}$

$$P(A \cup B) = \frac{16}{81} + \frac{25}{81}$$

$$= \frac{41}{81}$$

Example 60. *Marksmen A and B compete by taking turns to shoot a target. Odds in favour of A hitting the target (in a single try) are 3 : 2 and the odds in favour of B hitting the target (in a single try) are 4 : 3. Calculate the probability of A winning the competition if he gets the first chance to shoot.*

Solution. P(A) = Probability of A hitting $= \frac{3}{3+2} = \frac{3}{5}$,

P(B) = Probability of B hitting $= \frac{4}{4+3} = \frac{4}{7}$

$\therefore$ Probability of A winning

= P(A hits in first turn) + P(A fails, B fails, A hits)

+ P(A fails, B fails, A fails, B fails, A hits) + ...

$$= \frac{3}{5} + \left(\frac{2}{5}\right)\left(\frac{3}{7}\right)\left(\frac{3}{5}\right) + \left(\frac{2}{5}\right)\left(\frac{3}{7}\right)\left(\frac{2}{5}\right)\left(\frac{3}{7}\right)\left(\frac{3}{5}\right) + \ldots$$

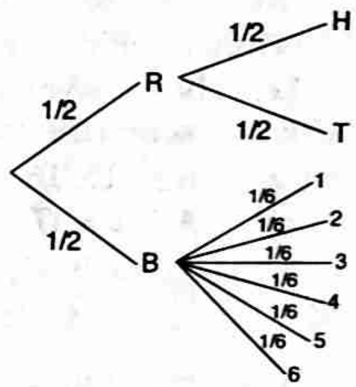
$$= \frac{3}{5}\left[1 + \frac{6}{35} + \left(\frac{6}{35}\right)^2 + \ldots\right] = \frac{3}{5}\left[\frac{1}{1-\frac{6}{35}}\right] = \frac{3}{5}\cdot\frac{35}{29} = \frac{21}{29}.$$

Example 61. *A ball is drawn from an urn containing one red ball and one black ball. If the ball drawn is red, a coin is tossed; if it is black, a die is thrown. What is the probability of*

(i) each outcome

(ii) getting a head

(iii) getting an even number?



Solution. (*i*) When the ball drawn is red (R) a coin is tossed. It may come up heads (H) or tails (T), and the outcomes are RH, RT. As the red ball can be drawn with equal probability as the black ball, $P(R) = P(B) = \frac{1}{2}$.

$\therefore$ $P(RH) = \frac{1}{2} \times \frac{1}{2} = \frac{1}{4}$, $P(RT) = \frac{1}{2} \times \frac{1}{2} = \frac{1}{4}$.

Similarly, when the ball drawn is black, a die is thrown, and the outcomes are B1, B2, B3, B4, B5, B6. The probability of each of these is $\frac{1}{2} \times \frac{1}{6} = \frac{1}{12}$ as shown by tree diagram also.

Hence, the sample space is S = {RH, RT, B1, B2, B3, B4, B5, B6}, and the respective probabilities of these outcomes are $\frac{1}{4}, \frac{1}{4}, \frac{1}{12}, \frac{1}{12}, \frac{1}{12}, \frac{1}{12}, \frac{1}{12}, \frac{1}{12}$.

(*ii*) The case favourable to the event "getting a head" is {RH} only, so required probability $= \frac{1}{4}$.

(*iii*) The cases favourable to the event "getting an even number" are {B2, B4, B6}, so required probability $= \frac{1}{12} + \frac{1}{12} + \frac{1}{12} = \frac{3}{12} = \frac{1}{4}$.

4 THE LAW OF TOTAL PROBABILITY

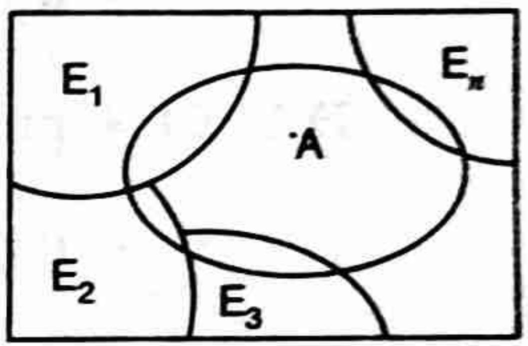
If $E_1, E_2, E_3, \ldots, E_n$ are mutually exclusive and exhaustive events associated with a sample space S of a random experiment and A is any event associated with S, then

$$P(A) = P(E_1)\, P(A \mid E_1) + P(E_2)\, P(A \mid E_2) + \ldots + P(E_n)\, P(A \mid E_n).$$

Proof. Since $E_1, E_2, \ldots, E_n$ are mutually exclusive and exhaustive events of S, therefore,

$$S = E_1 \cup E_2 \cup E_3 \ldots \cup E_n,$$

where $E_i \cap E_j = \phi$ for $i \neq j$.



Now $A = A \cap S = A \cap (E_1 \cup E_2 \cup E_3 \ldots \cup E_n)$

$= (A \cap E_1) \cup (A \cap E_2) \cup (A \cap E_3) \ldots (A \cap E_n)$ ...(*i*)

Also $A \cap E_i$ and $A \cap E_j$ are subsets of E_i and E_j respectively and it is known that $E_i \cap E_j = \phi$ for $i \neq j$, it follows that $A \cap E_i$ and $A \cap E_j$ are disjoint *i.e.* mutually exclusive for $i \neq j$, $i, j = 1, 2, 3, \ldots, n$.

From (*i*), we get

$$P(A) = P(A \cap E_1) + P(A \cap E_2) + P(A \cap E_3) + \ldots + P(A \cap E_n).$$

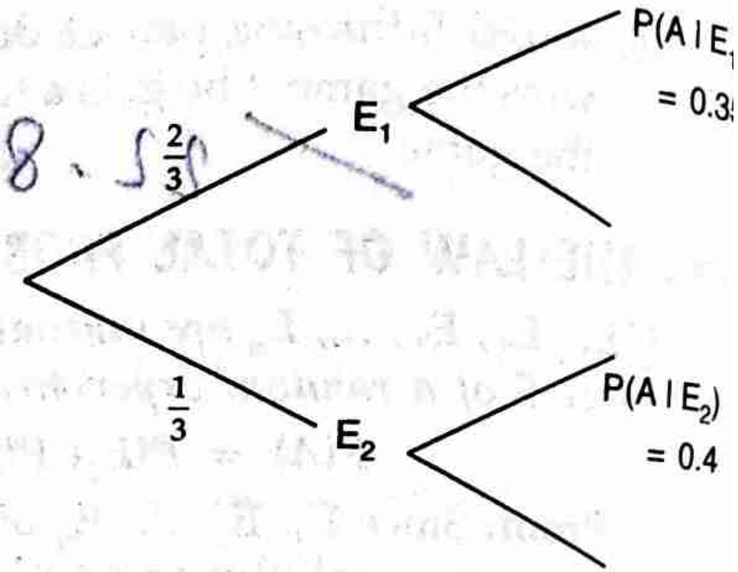
By using multiplication rule of probability, we get

$$P(A) = P(E_1)\, P(A \mid E_1) + P(E_2)\, P(A \mid E_2) + \ldots + P(E_n)\, P(A \mid E_n).$$

Example 2. *Two-thirds of the students of a class are boys and the rest are girls. It is known that th probability of a girl getting a first class marks in Council's Exam is 0.4 and a boy getting first clas marks is 0.35. Find the probability that a student chosen at random will get first class marks in Exam*

Solution. Let E_1, E_2 and A be the events defined as follows :

$$E_1 = \text{a boy is chosen,}$$
$$E_2 = \text{a girl is chosen and}$$
$$A = \text{the student gets first class marks.}$$



Then $P(E_1) = \frac{2}{3}$ and $P(E_2) = \frac{1}{3}$.

Note that E_1 and E_2 are mutually exclusive and exhaustive events.

$$P(A \mid E_1) = \text{probability of a boy getting first class marks}$$

$$= 0.35 = \frac{35}{100} = \frac{7}{20} \text{ and}$$

$$P(A \mid E_2) = \text{probability of a girl getting first class marks}$$

$$= 0.4 = \frac{4}{10} = \frac{2}{5}.$$

By using law of total probability, we get

$$P(A) = P(E_1)\,P(A \mid E_1) + P(E_2)\,P(A \mid E_2)$$

$$= \frac{2}{3}\cdot\frac{7}{20} + \frac{1}{3}\cdot\frac{2}{5} = \frac{7}{30} + \frac{2}{15} = \frac{11}{30}.$$

Example 6. *A fair die is rolled. If 1 turns up, a ball is picked up at random from bag A. If 2 or 3 turns up, a ball is picked up from bag B. If 4, 5 or 6 turns up, a ball is picked up from bag C. Bag A contains 3 red and 2 white balls; bag B contains 3 red and 4 white balls; bag C contains 4 red and 5 white balls. The die is rolled, a bag is selected and a ball is drawn. Find the probability that a red ball is drawn.*

Solution. Let E_1, E_2, E_3 and A be the events defined as follows :

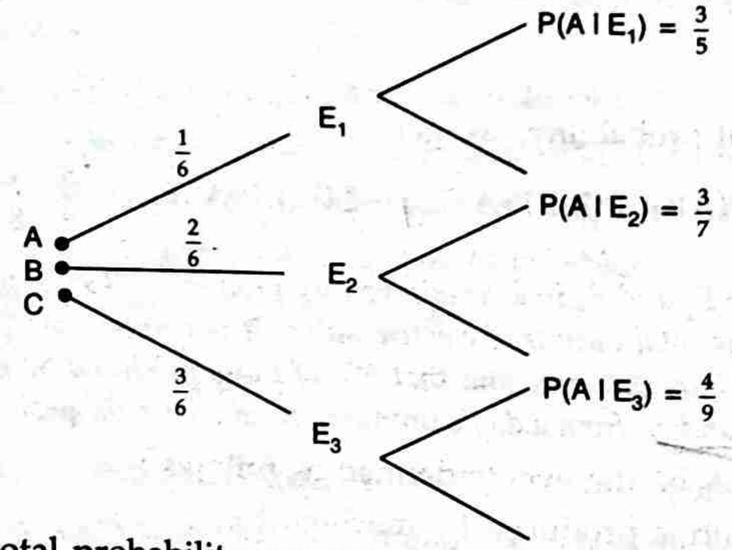
E_1 = bag A is picked up,
E_2 = bag B is picked up,
E_3 = bag C is picked up and
A = a red ball is drawn from the selected bag.

Then $P(E_1) = \frac{1}{6}, P(E_2) = \frac{2}{6}$ and $P(E_3) = \frac{3}{6}$.

Note that E_1, E_2 and E_3 are mutually exclusive and exhaustive events.

$P(A \mid E_1)$ = probability of drawing a red ball from bag A = $\frac{3}{5}$,

$P(A \mid E_2) = \frac{3}{7}$ and $P(A \mid E_3) = \frac{4}{9}$ (as shown in tree diagram)



By using law of total probability, we get

$$P(A) = P(E_1)\,P(A \mid E_1) + P(E_2)\,P(A \mid E_2) + P(E_3)\,P(A \mid E_3)$$

$$= \frac{1}{6}\cdot\frac{3}{5} + \frac{2}{6}\cdot\frac{3}{7} + \frac{3}{6}\cdot\frac{4}{9} = \frac{1}{10} + \frac{1}{7} + \frac{2}{9}$$

$$= \frac{63 + 90 + 140}{630} = \frac{293}{630}.$$

An urn contains m white balls and n black balls. A ball is drawn at random and is put back into the urn along with k additional balls of the same colour. A ball is again drawn at random. Show that the probability of drawing a white ball does not depend on k.

Solution. Let E_1, E_2 and A be the events defined as follows :

E_1 = white ball is drawn from the urn in first draw,

E_2 = black ball is drawn from the urn in first draw and

A = again white ball is drawn from the urn in second draw.

As the urn contains m white and n black balls,

$$P(E_1) = \frac{m}{m+n} \text{ and } P(E_2) = \frac{n}{m+n}.$$

Note that E_1 and E_2 are mutually exclusive and exhaustive events.

$P(A|E_1)$ = probability of drawing a white ball from the urn in second draw after putting k additional white balls into the urn

$$= \frac{m+k}{m+n+k} \text{ and}$$

$P(A|E_2)$ = probability of drawing a white ball from the urn in second draw after putting k additional black balls into the urn

$$= \frac{m}{m+n+k}.$$

Using the law of total probability, we get

$$P(A) = P(E_1)\,P(A|E_1) + P(E_2)\,P(A|E_2)$$

$$= \frac{m}{m+n}\cdot\frac{m+k}{m+n+k} + \frac{n}{m+n}\cdot\frac{m}{m+n+k}$$

$$= \frac{m^2+mk+mn}{(m+n)(m+n+k)} = \frac{m(m+n+k)}{(m+n)(m+n+k)}$$

$$= \frac{m}{m+n} = P(E_1).$$

Hence, the probability of drawing a white ball from the urn does not depend on k.

10.5 BAYE'S THEOREM

If $E_1, E_2, E_3, \ldots, E_n$ are mutually exclusive and exhaustive events associated with a random experiment and A is any event associated with the experiment, then

$$P(E_i \mid A) = \frac{P(E_i)\, P(A \mid E_i)}{\Sigma P(E_i)\, P(A \mid E_i)}, \text{ where } i = 1, 2, 3, \ldots, n.$$

Proof. By the law of total probability, we have

$$P(A) = P(E_1)\, P(A \mid E_1) + P(E_2)\, P(A \mid E_2) + \ldots + P(E_n)\, P(A \mid E_n)$$

$$= \Sigma P(E_i)\, P(A \mid E_i) \qquad \ldots(i)$$

Also by multiplication law of probability, we have

$$P(A \cap E_i) = P(A)\, P(E_i \mid A) = P(E_i)\, P(A \mid E_i),\ i = 1, 2, 3, \ldots, n$$

$$\Rightarrow \quad P(E_i \mid A) = \frac{P(E_i)\, P(A \mid E_i)}{P(A)},\ i = 1, 2, 3, \ldots, n$$

$$\Rightarrow \quad P(E_i \mid A) = \frac{P(E_i)\, P(A \mid E_i)}{\Sigma P(E_i)\, P(A \mid E_i)},\ i = 1, 2, 3, \ldots, n \qquad \text{(by using (i))}$$

The probability $P(E_i \mid A)$ means finding the probability of event E_i given that event A has occurred. Probability $P(E_i)$ was already known — so it was ***a priori probability.*** $P(E_i \mid A)$ is to be calculated after the knowledge that event A has happened — so it is called ***posteriori probability***.

For example, suppose that in a factory, 60% product are manufactured by machine M_1 and 40% by machine M_2. Machine M_1 produces 1% defective items and machine M_2 produces 2% defective items, and let

E_1 = event that product is manufactured by machine M_1,

E_2 = event that product is manufactured by machine M_2 and

A = event that product is defective.

Then from given information, we have

$$P(E_1) = 0.60,\ P(E_2) = 0.40,\ P(A \mid E_1) = 0.01,\ P(A \mid E_2) = 0.02$$

From law of total probability, we can calculate that the probability of a product being defective,

$$P(A) = P(E_1) \,.\, P(A \mid E_1) + P(E_2) \,.\, P(A \mid E_2)$$
$$= (0.60)\ (0.01) + (0.40)\ (0.02) = 0.006 + 0.008 = 0.014$$

Thus, 1.4% products of the factory are defective.

If the event A happens *i.e.* if we pick up a product and find that it is defective, $P(E_1 \mid A)$ means finding the probability that it was manufactured by machine M_1. You may think that it is 60% as we are given that 60% products are manufactured by machine M_1. However, according to Bayes' theorem,

$$P(E_1 \mid A) = \frac{P(E_1).P(A/E_1)}{P(E_1).P(A/E_1) + P(E_2).P(A/E_2)}$$

$$= \frac{(0.60)(0.01)}{(0.60)(0.01) + (0.40)(0.02)} = \frac{0.006}{0.006 + 0.008} = \frac{3}{7}$$ *i.e.* 43% approximately.

Thus if we pick up a product, there is 60% chance that it came from machine M_1 and 40% chance that it came from machine M_2. However, when we examine the product and find it to be defective — we revise our probabilities (from $P(E_1)$ to $P(E_1 \mid A)$ etc.) and say that there is 43% chance that this product came from machine M_1 and 57% chance that it came from machine M_2. Thus, *a priori* probability of 60% is revised to *posteriori* probability 43% with the *additional information* that the product is defective. This is what real life is all about. When we receive additional information, we revise our first opinions.

In answering a question on a multiple choice test, a student either knows the answer or guesses. Let $\frac{3}{4}$ be the probability that he knows the answer. Assuming that the student who guesses at the answer will be correct with a probability $\frac{1}{4}$, what is the probability that he knew the answer given that he answered it correctly?

Solution. Let E_1, E_2 and A be the events defined as follows :

E_1 = student knows the answer,

E_2 = student guesses the answer and

A = the student has answered the question correctly.

Then $P(E_1) = \frac{3}{4}$, so $P(E_2) = 1 - \frac{3}{4} = \frac{1}{4}$.

$P(A \mid E_1)$ = P(student answered the question correctly given that he knows the answer)

$= 1$ and

$P(A \mid E_2)$ = P(student answered the question correctly given that he guesses the answer)

$= \frac{1}{4}$.

We want to find $P(E_1 \mid A)$ = probability that the selected student knew the answer correctly.

By Baye's theorem, we have

$$P(E_1 \mid A) = \frac{P(E_1)P(A \mid E_1)}{P(E_1)P(A \mid E_1) + P(E_2)P(A \mid E_2)}$$

$$= \frac{\frac{3}{4}.1}{\frac{3}{4}.1 + \frac{1}{4}.\frac{1}{4}} = \frac{3}{3 + \frac{1}{4}} = \frac{12}{13}.$$

There are two groups of subjects, one of which consists of 5 Science and 3 Engineering subjects; and the other consists of 3 Science and 5 Engineering subjects. An unbiased die is rolled. If number 1 or 6 turn up, a subject is selected at random from the first group, otherwise a subject is selected from the second group. If ultimately an Engineering subject is selected, then find the probability that it is selected from second group.

Solution. Let E_1, E_2 and A be the events defined as follows :

E_1 = die turns up with number 1 or 6 *i.e.* selecting first group of subjects

E_2 = die turns up with number 2, 3, 4 or 5 *i.e.* selecting second group of subjects and

A = Engineering subject is selected.

Then $P(E_1) = \frac{2}{6} = \frac{1}{3}$ and $P(E_2) = \frac{4}{6} = \frac{2}{3}$.

$P(A \mid E_1)$ = P(selecting Engineering subject from first group) $= \frac{3}{8}$,

$P(A \mid E_2)$ = P(selecting Engineering subject from second group) $= \frac{5}{8}$.

We want to find $P(E_2 \mid A)$.

By Baye's theorem, we have

$$P(E_2 \mid A) = \frac{P(E_2)\,P(A \mid E_2)}{P(E_1)P(A \mid E_1) + P(E_2)\,P(A \mid E_2)}$$

$$= \frac{\frac{2}{3}\cdot\frac{5}{8}}{\frac{1}{3}\cdot\frac{3}{8} + \frac{2}{3}\cdot\frac{5}{8}} = \frac{10}{13}.$$

Bag I contains 4 red and 5 black balls and bag II contains 3 red and 4 black balls. One ball is transferred from bag I to bag II and then two balls are drawn at random (without replacement) from bag II. The balls so drawn are both found to be black. Find the probability that the transferred ball was black.

Solution. Let E_1, E_2 and A be the events defined as follows :

E_1 = red ball is transferred from bag I to bag II,

E_2 = black ball is transferred from bag I to bag II and

A = two black balls has been drawn from bag II.

As the bag I contains 4 red and 5 black balls,

$$\therefore \quad P(E_1) = \frac{4}{9}, \; P(E_2) = \frac{5}{9}.$$

When E_1 has occurred *i.e.* when a red ball has been transferred from bag I to bag II, then bag II has 4 red and 4 black balls.

$\therefore$ $P(A|E_1)$ = probability of drawing 2 black balls from bag II when E_1 has occurred

$$= \frac{^4C_2}{^8C_2} = \frac{4.3}{1.2} \times \frac{1.2}{8.7} = \frac{3}{14}.$$

When E_2 has occurred *i.e.* when a black ball has been transferred from bag I to bag II, then bag II has 3 red and 5 black balls.

$\therefore$ $P(A|E_2)$ = probability of drawing 2 black balls from bag II when E_2 has occurred

$$= \frac{^5C_2}{^8C_2} = \frac{5.4}{1.2} \times \frac{1.2}{8.7} = \frac{5}{14}.$$

We want to find $P(E_2|A)$.

By Baye's theorem, we get

$$P(E_2|A) = \frac{P(E_2)\,P(A|E_2)}{P(E_1)\,P(A|E_1) + P(E_2)\,P(A|E_2)}$$

$$= \frac{\frac{5}{9}\cdot\frac{5}{14}}{\frac{4}{9}\cdot\frac{3}{14} + \frac{5}{9}\cdot\frac{5}{14}} = \frac{25}{12+25} = \frac{25}{37}.$$

Coloured balls are distributed in four boxes as follows :

Box	*Colour*			
	Black	*White*	*Red*	*Blue*
I	3	4	5	6
II	2	2	2	2
III	1	2	3	1
IV	4	3	1	5

A box is selected at random and a ball is drawn. If the colour of the ball is black, what is the probability that ball drawn is from the box III?

Solution. Let E_1, E_2, E_3, E_4 and A be the events defined as follows :

E_1 = box I is chosen,

E_2 = box II is chosen,

E_3 = box III is chosen,

E_4 = box IV is chosen and

A = ball draw is black.

As a box is selected at random,

$$\therefore \quad P(E_1) = P(E_2) = P(E_3) = P(E_4) = \frac{1}{4}.$$

Box I contains 3 black, 4 white, 5 red and 6 blue balls, so the total number of balls in box I = 3 + 4 + 5 + 6 = 18.

$P(A \mid E_1)$ = probability of drawing a black ball when E_1 has occurred *i.e.* drawing a black ball from box I

$$= \frac{3}{18} = \frac{1}{6}.$$

Similarly, $P(A \mid E_2) = \frac{2}{8} = \frac{1}{4}$, $P(A \mid E_3) = \frac{1}{7}$ and $P(A \mid E_4) = \frac{4}{13}$.

By using law of total probability,

$$P(A) = \Sigma P(E_i)\, P(A \mid E_i) = \frac{1}{4}\cdot\frac{1}{6} + \frac{1}{4}\cdot\frac{1}{4} + \frac{1}{4}\cdot\frac{1}{7} + \frac{1}{4}\cdot\frac{4}{13}$$

$$= \frac{1}{4}\left(\frac{1}{6} + \frac{1}{4} + \frac{1}{7} + \frac{4}{13}\right) = \frac{1}{4}\cdot\frac{182 + 273 + 156 + 336}{12 \times 91} = \frac{947}{48 \times 91}.$$

We want to find $P(E_3 \mid A)$.

By Baye's theorem, we have :

$$P(E_3 \mid A) = \frac{P(E_3)\, P(A \mid E_3)}{P(A)} = \frac{\frac{1}{4}\cdot\frac{1}{7}}{\frac{947}{48 \times 91}} = \frac{1}{28} \times \frac{48 \times 91}{947} = \frac{156}{947}.$$

Example. *A man is known to speak truth 3 out of 4 times. He throws a die and reports that it is a six. Find the probability that it is actually a six.*

Solution. Let E_1, E_2 and A be the events defined as follows :

E_1 = die shows six *i.e.* six has occurred,

E_2 = die does not show six *i.e.* six has not occurred and

A = the man reports that six has occurred.

We wish to calculate the probability that six has actually occurred given that the man reports that six occurs *i.e.* $P(E_1 | A)$.

Now, $P(E_1) = \frac{1}{6}$, $P(E_2) = \frac{5}{6}$,

$P(A | E_1)$ = probability that the man reports that six occurs given that six has occurred

= probability that the man is speaking the truth $= \frac{3}{4}$ and

$P(A | E_2)$ = probability that the man reports that six occurs given that six has not occurred

= probability that the man does not speak truth $= \frac{1}{4}$.

By Bayes' theorem, we have :

$$P(E_1 | A) = \frac{P(E_1)\,P(A | E_1)}{P(E_1)\,P(A | E_1) + P(E_2)\,P(A | E_2)}$$

$$= \frac{\frac{1}{6}\cdot\frac{3}{4}}{\frac{1}{6}\cdot\frac{3}{4} + \frac{5}{6}\cdot\frac{1}{4}} = \frac{3}{8}.$$

A card from a pack of 52 cards is lost. From the remaining cards of the pack, two cards are drawn and are found to be both clubs. Find the probability of the lost card being a club.

Solution. Let E_1, E_2 and A be the events defined as follows :

E_1 = lost card is of clubs,

E_2 = lost card is not of clubs and

A = two cards drawn are both of clubs.

Then $P(E_1) = \frac{13}{52} = \frac{1}{4}$ and $P(E_2) = \frac{39}{52} = \frac{3}{4}$.

When one card is lost, number of remaining cards in the pack = 51.

When E_1 has occured *i.e.* a card of clubs is lost, then the probability of drawing 2 cards of clubs from the remaining pack $= \frac{^{12}C_2}{^{51}C_2}$,

so $$P(A \mid E_1) = \frac{^{12}C_2}{^{51}C_2} = \frac{12.11}{1.2} \times \frac{1.2}{51.50} = \frac{22}{425}.$$

When E_2 has occurred *i.e.* when a card of clubs is not lost, then the probability of drawing 2 cards of clubs from the remaining pack $= \frac{^{13}C_2}{^{51}C_2}$,

so $$P(A \mid E_2) = \frac{^{13}C_2}{^{51}C_2} = \frac{13.12}{1.2} \times \frac{1.2}{51.50} = \frac{26}{425}.$$

We want to find $P(E_1/A)$.

By Baye's theorem, we have :

$$P(E_1 \mid A) = \frac{P(E_1)P(A \mid E_1)}{P(E_1)P(A \mid E_1) + P(E_2)P(A \mid E_2)}$$

$$= \frac{\frac{1}{4} \cdot \frac{22}{425}}{\frac{1}{4} \cdot \frac{22}{425} + \frac{3}{4} \cdot \frac{26}{425}} = \frac{22}{22+78} = \frac{22}{100} = \frac{11}{50}.$$

Random Variables and Probability Distributions

2.1 RANDOM VARIABLES

Suppose that to each point of a sample space we assign a number. We then have a *function* defined on the sample space. This function is called a *random variable* (or *stochastic variable*) or more precisely a *random function* (*stochastic function*). It is usually denoted by a capital letter such as X or Y. In general, a random variable has some specified physical, geometrical, or other significance.

Example 2.1 Suppose that a coin is tossed twice so that the sample space is $S = \{HH, HT, TH, TT\}$. Let X represent the number of heads that can come up. With each sample point we can associate a number for X as shown in Table 2.1. Thus, for example, in the case of HH (i.e., 2 heads), $X = 2$ while for TH (1 head), $X = 1$. It follows that X is a random variable.

It should be noted that many other random variables could also be defined on this sample space, for example, the square of the number of heads or the number of heads minus the number of tails.

Table 2.1

Sample Point	*HH*	*HT*	*TH*	*TT*
X	2	1	1	0

A random variable that takes on a finite or countably infinite number of values (see page 4) is called a *discrete random variable* while one which takes on a noncountably infinite number of values is called a *nondiscrete random variable*.

2.2 DISCRETE PROBABILITY DISTRIBUTIONS

Let X be a discrete random variable, and suppose that the possible values that it can assume are given by $x_1, x_2, x_3, \ldots$, arranged in some order. Suppose also that these values are assumed with probabilities given by

$$P(X = x_k) = f(x_k) \qquad k = 1, 2, \ldots \tag{1}$$

It is convenient to introduce the *probability function*, also referred to as *probability distribution*, given by

$$P(X = x) = f(x) \tag{2}$$

For $x = x_k$, this reduces to (1) while for other values of x, $f(x) = 0$.

In general, $f(x)$ is a probability function if

1. $f(x) \geq 0$

2. $\sum f(x) = 1$

where the sum in 2 is taken over all possible values of x.

Example 2.2 Find the probability function corresponding to the random variable X of Example 2.1. Assuming that the coin is fair, we have

$$P(HH) = \frac{1}{4} \qquad P(HT) = \frac{1}{4} \qquad P(TH) = \frac{1}{4} \qquad P(TT) = \frac{1}{4}$$

Then

$$P(X = 0) = P(TT) = \frac{1}{4}$$

$$P(X = 1) = P(HT \cup TH) = P(HT) + P(TH) = \frac{1}{4} + \frac{1}{4} = \frac{1}{2}$$

$$P(X = 2) = P(HH) = \frac{1}{4}$$

The probability function is thus given by Table 2.2.

Table 2.2

x	0	1	2
$f(x)$	1/4	1/2	1/4

2.3 DISTRIBUTION FUNCTIONS FOR RANDOM VARIABLES

The *cumulative distribution function*, or briefly the *distribution function*, for a random variable X is defined by

$$F(x) = P(X \le x) \tag{3}$$

where x is any real number, i.e., $-\infty < x < \infty$.

The distribution function $F(x)$ has the following properties:

1. $F(x)$ is nondecreasing [i.e., $F(x) \le F(y)$ if $x \le y$].
2. $\lim_{x \to -\infty} F(x) = 0$; $\lim_{x \to \infty} F(x) = 1$.
3. $F(x)$ is continuous from the right [i.e., $\lim_{h \to 0^+} F(x + h) = F(x)$ for all x].

2.4 DISTRIBUTION FUNCTIONS FOR DISCRETE RANDOM VARIABLES

The distribution function for a discrete random variable X can be obtained from its probability function by noting that, for all x in $(-\infty, \infty)$,

$$F(x) = P(X \le x) = \sum_{u \le x} f(u) \tag{4}$$

where the sum is taken over all values u taken on by X for which $u \le x$.

If X takes on only a finite number of values $x_1, x_2, \ldots, x_n$, then the distribution function is given by

$$F(x) = \begin{cases} 0 & -\infty < x < x_1 \\ f(x_1) & x_1 \le x < x_2 \\ f(x_1) + f(x_2) & x_2 \le x < x_3 \\ \vdots & \vdots \\ f(x_1) + \cdots + f(x_n) & x_n \le x < \infty \end{cases} \tag{5}$$

Example 2.3 (a) Find the distribution function for the random variable X of Example 2.2. (b) Obtain its graph.

(a) The distribution function is

$$F(x) = \begin{cases} 0 & -\infty < x < 0 \\ \frac{1}{4} & 0 \le x < 1 \\ \frac{3}{4} & 1 \le x < 2 \\ 1 & 2 \le x < \infty \end{cases}$$

(b) The graph of $F(x)$ is shown in Fig. 2.1.

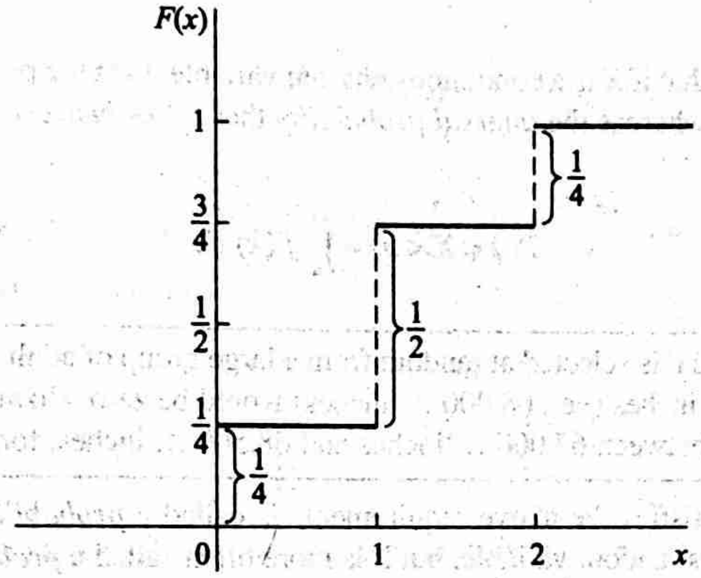


Fig. 2.1

The following things about the above distribution function, which are true in general, should be noted.

1. The magnitudes of the jumps at 0, 1, 2 are $\frac{1}{4}, \frac{1}{2}, \frac{1}{4}$ which are precisely the probabilities in Table 2.2. This fact enables one to obtain the probability function from the distribution function.
2. Because of the appearance of the graph of Fig. 2.1, it is often called a *staircase function* or *step function*. The value of the function at an integer is obtained from the higher step; thus the value at 1 is $\frac{3}{4}$ and not $\frac{1}{4}$. This is expressed mathematically by stating that the distribution function is *continuous from the right* at 0, 1, 2.
3. As we proceed from left to right (i.e. going *upstairs*), the distribution function either remains the same or increases, taking on values from 0 to 1. Because of this, it is said to be a *monotonically increasing function*.

It is clear from the above remarks and the properties of distribution functions that the probability function of a discrete random variable can be obtained from the distribution function by noting that

$$f(x) = F(x) - \lim_{u \to x^-} F(u) \tag{6}$$